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The geometry of the Hill equation and of the Neumann system

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Let there be given a finite-gap operator $\mathrm{L} = \mathrm{d}^2/\mathrm{d}x^2 + q$ and its Baker function $\psi(x, p)$, which is analytic for p on a certain hyperelliptic curve C. It is shown that a sequence of Bäcklund transformations maps C to a projective space. This embedding can be interpreted as a matrix representation of the Hill equation by the Neumann system of constrained harmonic oscillators. The image curve, C', lies on a rational ruled surface; the structure of this surface is explained by use of ideas due to Burchnall & Chaundy (*Proc. R. Soc. Lond.* A **118**, 557–583 (1928)). Baker functions and Bäcklund transformations are then used to define a (many-to-many) correspondence between effective divisors on the curve C and points lying on a quadric, or in the intersection of two or more quadrics. This relates the theory of the Hill equation to earlier work of Knörrer, Moser and Reid. It is then shown that the Kummer image of the Jacobian of C can be realized as a hypersurface in the space of momentum variables of the Neumann system. Further projects, such as extensions to non-hyperelliptic curves, are outlined.

1. Introduction

A differential operator $\mathrm{L} = \mathrm{D}^2 + q(x)$, $\mathrm{D} = \mathrm{d}/\mathrm{d}x$, is said to be 'finite-gap' if it commutes with a differential operator B of odd order, $[\mathrm{L}, \mathrm{B}] = 0$. A 'finite-gap potential' $q(x)$ is therefore a time-independent, or stationary, solution of an equation $\partial \mathrm{L}/\partial t = [\mathrm{B}, \mathrm{L}]$ in the Korteweg–de Vries hierarchy. Because L and B commute, they have a common eigenfunction

$$\mathrm{L}\psi = E\psi,$$
$$\mathrm{B}\psi = R\psi.$$

The eigenvalues E, R are known to be related by an algebraic equation

$$R^2 = \prod_{j=1}^{2j+1} (E - \epsilon_j), \tag{1}$$

and the common eigenfunction ψ (the 'Baker function') is an analytic function on the Riemann surface (1) or, equivalently, a holomorphic section of a certain line bundle on (1). Until now, the theory of finite-gap operators has drawn mostly on the analytical aspects of Riemann surfaces and on their abstract, intrinsic geometry.

Our aim in this paper is to explain some of the *extrinsic* properties of the curves, line bundles, and isospectral tori (Jacobians) when those are embedded as concrete objects in a projective space.

There are several reasons for studying geometric realizations of the finite-gap operator theory. The classical theory of curves and Jacobians is very beautiful, and an interesting statement about abstract curves and line bundles should be worth repeating about concrete representations. Furthermore, when one integrable system, like the stationary Lax equation $[\mathrm{L}, \mathrm{B}] = 0$, is

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mapped to a projective space, a new integrable system may result. We show below, for example, that the Neumann system of constrained harmonic oscillators is a matrix representation of $[\mathrm{L},\mathrm{B}]=0$. Finally, the Baker function provides a way to get geometric conclusions via transcendental methods. This idea seems, to us, to be promising. We use it to construct an analytically tractable embedding of the curve C (and its Baker function) into projective space. We then study the geometric properties of the embedded curve. It lies on a rational normal scroll, which, we believe, has an interesting relation to the Kummer map (by quadratic θ-functions) of the Jacobian of C. The Weil–Reid Jacobian of $(g-1)$-planes in the intersection of two quadrics in $\mathbb{P}^{2g+1}$ enters the picture, and we clarify some of its connections with the Neumann system. As a sample result of this approach, we show that the Kummer image of a hyperelliptic Jacobian can be realized as a hypersurface in the momentum space of the Neumann system.

This paper is a progress report, and tries to convey the flavour of our approach without overburdening the exposition with details. We hope to make the geometry interesting to soliton experts familiar with the analytic aspects of finite-gap potentials, and also to persuade geometers that soliton methods can be a useful supplement to the more standard θ-function calculations.

2. Baker functions

In this section we review the basic facts about Baker functions, primarily to fix our notation. The language of line bundles is probably more congenial to geometers, while a description via poles and exponential behaviour is customary in soliton theory. We will use both, and hope that the reader can follow the discussion by skipping unfamiliar terminology. Nothing in this section is new. We strongly recommend the beautiful papers by Burchnall & Chaundy (1922, 1928) and Baker (1928), which are rich in ideas but have come to be appreciated only after being rediscovered in the 1970s. See Krichever (1977) and Mumford (1978) for the modern perspective.

We fix a finite-gap operator $\mathrm{L}=\mathrm{D}^2+q(x)$ $(\mathrm{D}=\mathrm{d}/\mathrm{d}x)$. Such an L commutes with a differential operator B of odd order $2g+1$. It is known that there is then a common eigenfunction $\psi(x,p)$ of $\mathrm{L}\psi=E\psi$, $\mathrm{B}\psi=R\psi$, parametrized by the points $p=(E,R)$ of the Riemann surface of

$$C\colon R^2=\prod_{j=1}^{2g+1}(E-\epsilon_j). \tag{1}$$

We require that the branch points ϵ_j be distinct; this excludes soliton or rational potentials $q(x)$, but does admit the periodic and quasiperiodic ones.

The common eigenfunction of $\mathrm{L}\psi=E\psi$, $\mathrm{B}\psi=R\psi$ is determined up to multiplication by an arbitrary function of p. A unique ψ is picked out by the following normalization. Let $\delta=\delta_1+\ldots+\delta_g$ be a *non-special divisor* of degree g on C (i.e. the only meromorphic functions with poles only in δ are constants). There exists exactly one function $\psi(x,p)$, $x\in\mathbb{C}$, $p\in C$, with these properties:

(i) for fixed x sufficiently close to 0, $p\mapsto\psi(x,p)$ is meromorphic on $C-\infty$, with poles, *independent of* x, in δ;

(ii) $\psi(0,p)\equiv 1$;

(iii) near ∞, $\psi(x,p)\,\mathrm{e}^{-kx}=1+O(1/k)$, a holomorphic function, where $k=\sqrt{E}$;

(iv) for fixed $p\in C-\infty-\delta$, $x\mapsto\psi(x,p)$ is holomorphic in a neighbourhood of $0\in\mathbb{C}$.

Such a function will be called a *Baker function*. If L is a finite-gap operator, the common eigenfunction of $L\psi = E\psi$, $B\psi = R\psi$, may be normalized to be a Baker function, and conversely, any Baker function determines a unique finite-gap operator $L = D^2 + q(x)$. It turns out that, while the poles $\delta_1, \ldots, \delta_g$ of ψ are independent of x, the zeros $\mu_1(x), \ldots, \mu_g(x)$ vary with x; they satisfy a linearizable system of differential equations $\mu_j'(x) = f_j(\mu_1, \ldots, \mu_g)$ whose solution determines $q(x)$, see (McKean & van Moerbeke 1975).

In more geometric language, ψ is the function on C corresponding to a certain holomorphic section of a line bundle over C. Let $\mathscr{L}(\delta)$ be the usual line bundle of the divisor δ, and let $\mathscr{L}(e^{kx})$ be the line bundle with transition function e^{-kx} into a neighbourhood of ∞. Then $\mathscr{L}(x) \underset{\text{def}}{=} \mathscr{L}(\delta) \otimes \mathscr{L}(e^{kx})$ has (by non-speciality of δ) a one-dimensional space of holomorphic sections. The function ψ is defined by the section σ normalized to $\sigma(\infty) = 1$. One can check that $\mathscr{L}(x)$ has Chern class g, and that $\mathscr{L}(x) \cong \mathscr{L}(\mu(x))$, where $\mu(x)$ is the zero divisor of ψ.

In this way, all information about a finite-gap potential $q(x)$ is encoded in the properties of the analytic function $\psi(x, p)$ (or, in the properties of the family $\mathscr{L}(x)$ of line bundles on C, if one prefers to express it that way). We will concentrate on Baker functions and their line bundles, and say very little about the potentials $q(x)$.

3. Isospectral tori

It is known that the set of (complex) potentials $q(x)$ associated to a given curve (1) is parametrized by a (complex) torus, sometimes called the 'isospectral torus', which corresponds to the set of possible poles $\delta_1 + \ldots + \delta_g$ of Baker functions on C. Some singular potentials, having a pole at $x = 0$, have to be put in; they arise from the special divisors δ. The precise description of the isospectral torus is as follows.

Two divisors β, β' on C are called (*linearly*) *equivalent* if there is a meromorphic function whose divisor of zeros and poles is $\beta - \beta'$. The set of equivalence classes of divisors of degree d is an algebraic variety, the *Picard variety of* C, $Pic^{(d)}(C)$. The pole divisors $\delta = \delta_1 + \ldots + \delta_g$ of the Baker function range over $Pic^{(g)}(C)$.

The Picard varieties are all isomorphic to a complex torus, the Jacobian of C. Thanks to this isomorphism, $Pic^{(g)}(C)$ can be viewed as torus, which is referred to as the 'isospectral torus'. It is important, however, to realize that there are many different ways of identifying $Pic^{(g)}(C)$ with $Jac(C)$, and some of the geometry becomes clear only when such identifications are kept track of rather carefully.

To show that $Pic^{(g)}(C) \cong Jac(C)$, one uses the Abel map. Let $a_1, \ldots, a_g$, $b_1, \ldots, b_g$ be a basis of homology cycles, with $a_i \cdot a_j = b_i \cdot b_j = 0$, $a_i \cdot b_j = \delta_{ij}$. Let $\omega_1, \ldots, \omega_g$ be a normalized basis of holomorphic differentials, i.e. $\oint_{a_i} \omega_j = \delta_{ij}$. Fix $\delta^0 \in Pic^{(g)}(C)$. The Abel map with base point δ^0,

$$A : \delta \rightarrow \left(\sum_{j=1}^{g} \int_{\delta_j^0}^{\delta_j} \omega_1, \ldots, \sum_{j=1}^{g} \int_{\delta_j^0}^{\delta_j} \omega_g \right)$$

takes $Pic^{(g)}(C)$ to the torus

$$Jac(C) \underset{\text{def}}{=} C^g / \Lambda,$$

where Λ is the lattice of periods of the ω_j. *The isomorphism depends on the choice of basepoint* δ^0; we take δ^0 to be $\infty + \ldots + \infty$ from now on.

The Abel map is defined similarly for divisors in $Pic^{(d)}(C)$; the sum is then $\sum_{j=1}^{d}$. In particular, for points $p \in C \subset Pic^{(1)}(C)$,

$$A(p) = \left(\int_{\infty}^{p} \omega_1, \ldots, \int_{\infty}^{p} \omega_g\right) \in C^g/\Lambda. \tag{2}$$

This map embeds the curve C into its Jacobian.

4. Maps into projective space

In this section we give a brief introduction to embeddings of algebraic varieties into projective spaces. We hope that this may be of use to soliton analysts who are not familiar with some of the geometric ideas. Other readers may prefer to pass on to the next section.

First, recall that projective space $\mathbb{P}^r$ is the set of lines through the origin in $\mathbb{C}^{r+1}$. A line is given parametrically by the equations $u_j = a_i t$, $t \in \mathbb{C}$, $j = 0, \ldots, r$, with not all $a_j = 0$. The equations $u_j = \lambda a_j t$ describe the same line. The numbers $a_0, \ldots, a_r$ therefore determine a point in $\mathbb{P}^r$, and $\lambda a_0, \ldots, \lambda a_r$ determine the same point. These so-called *homogeneous coordinates* are written $[a_0 : \ldots : a_r]$.

We want to map the following complex manifolds to a projective space: (i) the curve C, (ii) the projective plane $\mathbb{P}^2$ in which the standard model $R^2 = \Pi(E - \epsilon_j)$ of C is situated, (iii) the Jacobian of C. Let M denote one of these manifolds. The goal, in all cases, is to represent the image of M by parametric equations: if $[z_0 : \ldots : z_r]$ are the homogeneous coordinates in $\mathbb{P}^r$, we want to map

$$M \ni m \mapsto z_0 = f_0(m), \ldots, z_r = f_r(m).$$

To this end, one must find a suitable set $\{f_0, \ldots, f_r\}$ of meromorphic functions on M (there are no non-constant *holomorphic* functions on compact complex manifolds). The standard way to generate such functions is as follows.

A *divisor* Δ on the curve (or Riemann surface) C is a formal sum of points, $\Delta = p_1 + \ldots + p_n - q_1 - \ldots - q_m$, interpreted as the potential set of poles and zeros of a meromorphic function on C. If M has (complex) dimension greater than 1, the set of poles and zeros of a meromorphic function will have dimension one less than M. Accordingly, a divisor Δ on M is a formal sum $\Delta = P_1 + \ldots + P_n - Q_1 - \ldots - Q_m$, where each P_i, Q_i is an irreducible algebraic subvariety of codimension 1. (If $M = \mathbb{P}^2$, the projective plane, for example, the P_i, Q_i would be algebraic curves.)

For a divisor $\Delta = P_1 + \ldots + P_n - Q_i - \ldots - Q_m$, introduce the linear space

$$L(\Delta) = \{\text{all meromorphic functions on } M, \text{ with poles at most in } P_1, \ldots, P_n \text{ and zeros at least in } Q_1, \ldots, Q_m\}.$$

Now let $\{f_0, \ldots, f_r\}$ be a basis of $L(\Delta)$, and try to define the map $i_\Delta : M \to \mathbb{P}^r$,

$$i_\Delta : m \mapsto [f_0(m) : \ldots : f_r(m)].$$

If all f_j have common zeros, i_Δ will be undefined at some points $m \in M$, since $[0 : \ldots : 0]$ is not an allowed set of homogeneous coordinates. In that situation, it may be necessary to 'blow up' the manifold M, as we will see later. Even if i_Δ is well defined on all of M, one must still check other properties of i_Δ. We will comment on the details later, as necessary.

One more geometrical idea should be introduced at this point. Consider the hyperplane

$\alpha_0 z_0 + \ldots + \alpha_r z_r = 0$ in $\mathbb{P}^r$. It intersects the image $i_\Delta(M)$ in the set where $\alpha_0 f_0(m) + \ldots + \alpha_r f_r(m) = 0$, that is, in the zero locus of the function $\alpha_0 f_0 + \ldots + \alpha_r f_r \in L(\Delta)$. There is, consequently, a correspondence between the zeros of functions in $L(\Delta)$ and hyperplanes in $\mathbb{P}^r$, which provides a first bridge between the analyst's language and the geometer's pictures. The set of hyperplanes in $\mathbb{P}^r$ and, equivalently, the linear space $L(\Delta)$, are called the (complete) *linear system of* Δ. A divisor *linearly equivalent* to the original Δ is simply a set cut out on $i_\Delta(M)$ by a hyperplane in $\mathbb{P}^r$: linear equivalence of divisors corresponds to linear transformation of hyperplanes.

5. Transferences

Consider now the curve $C: R^2 = \Pi(E - \epsilon_j)$ and a fixed Baker function on C. How can one map C into a larger projective space? According to §4, one must find a linear system on C. There are, of course, many candidates; we want the system to be built in some way from the Baker function, so that the projective embedding will encode the Baker function $\psi(x, p)$.

Recall (cf. §2) that the only *meromorphic* functions with poles among the poles $\delta_1 + \ldots + \delta_g \underset{\text{def}}{=} \delta$ of ψ are constants; that is, $L(\delta) = \mathbb{C}$. The one-dimensional linear system $L(\delta)$ will therefore not provide a useful projective embedding: all of the curve C maps to a single point. One must allow more poles than just δ, say $\delta_1 + \ldots + \delta_g + z_1 + \ldots + z_m \underset{\text{def}}{=} \delta + Z$, for there will be more functions satisfying less restrictive conditions. The poles Z will be independent of x. In the language of line bundles, we are changing $\mathscr{L}(\delta) \otimes \mathscr{L}(e^{kx})$ to $\mathscr{L}(\delta) \otimes \mathscr{L}(e^{kx}) \otimes \mathscr{L}(Z)$, or equivalently (see §2), $\mathscr{L}(\mu(x))$ to $\mathscr{L}(\mu(x)) \otimes \mathscr{L}(Z)$. At the level of divisors, the operation is a simple one. The divisor motion $x \to \mu(x)$ in $Pic^{(g)}(C)$ encodes the x-behaviour of the finite-gap potential q; we are now mapping this to a curve $x \to \mu(x) + Z$ in $Pic^{(g+m)}(C)$, which is clearly not very different. Still, as we will explain, the concrete analytic and geometric realizations of this apparently minor change are very different from the original $D^2 + q$.

As a first step, we show in this section how one can construct functions on C that have poles in $\delta + Z$ and the e^{kx} behaviour at ∞ (characteristic of Baker functions). In other words, we will build sections of $\mathscr{L}(\delta) \otimes \mathscr{L}(e^{kx}) \otimes \mathscr{L}(Z)$ from the unique section of $\mathscr{L}(\delta) \otimes \mathscr{L}(e^{kx})$. The method is a variation on the Burchnall & Chaundy (1928) concept of *transference*, which, in slightly different contexts, is also known as Bäcklund-, Crum-, or Darboux-transformation.

Proposition 1. *Let ψ be the Baker function with pole divisor δ, and let $p_* \in C - \delta - \infty$. Let $W(f, g) = f'g - fg'$ be the usual Wronskian of functions of x. Let τ denote the hyperelliptic involution, $\tau: (E, R) \to (E, -R)$.*

(i)
$$\chi(x, p) \underset{\text{def}}{=} \frac{W(\psi(x, p), \psi(x, \tau p_*))}{E(p) - E(p_*)} \tag{3}$$

has poles $\delta + p_*$, and goes like $\psi(x, \tau p_*) k^{-1} e^{kx}$ at ∞.

(ii) *Let $\tilde{\chi}(x, p) = \chi(x, p) / \psi(x, \tau p_*)$. Then*

$$\tilde{\psi}(x, p) \underset{\text{def}}{=} \tilde{\chi}(x, p) / \tilde{\chi}(0, p) \tag{4}$$

is again a Baker function.

(iii) *The potentials $q(x), \tilde{q}(x)$ corresponding to $\psi, \tilde{\psi}$ are related by*

$$\tilde{q}(x) = q(x) + 2 \frac{d^2}{dx^2} \ln \psi(x, p_*).$$

The proof is an easy calculation. We call the map $\psi \to \chi$ a *partial transference at* p_*, and the map $\psi \to \tilde{\psi}$ a *transference at* p_*.

PROPOSITION 2. *If* $\delta, \tilde{\delta}$ *are the pole divisors of* ψ *and of* $\tilde{\psi}$ *(obtained by transference at* p_**), then*

$$A(\tilde{\delta}) = A(\delta) + A(p_*).$$

The proof may be found in Burchnall & Chaundy (1928). This proposition shows that translation by $A(p_*)$ in the Jacobian can be expressed analytically as transference. More generally, Burchnall & Chaundy (1928) show the next proposition.

PROPOSITION 3. *Any two divisors* $\delta, \delta' \in Pic^{(g)}(C)$ *can be related by at most g transferences:*

$$A(\delta') = A(\delta) + A(p_1) + \ldots + A(p_g).$$

We will have several occasions to refer to these propositions, so a general explanation of their significance may suffice for now.

(1) Algebraic problems are solved by transcendental methods. For example, to find a meromorphic function with poles $\delta + p_*$, take the Baker function ψ corresponding to δ, form the partial transference χ of (3), and set $x = 0$. This is not claimed to be a computationally effective procedure. Rather, the intervention of a deformation in x results in formulas that are, in some circumstances, better than a mere existence theorem or a possibly very messy algebraic prescription.

(2) The Jacobi inversion method is put into a more tractable form (for our purposes). Suppose one wants to convert translation on the Jacobian into an operation on divisors: given $\delta \in Pic^{(g)}(C)$, $a \in \mathbb{C}^g$, for what δ' is $A(\delta') \equiv A(\delta) + a$? Jacobi inversion tells one to find zeros of a certain θ-function. Transference tells one to form Wronskians of Baker functions, and this prescription is more useful in certain soliton problems.

6. THE NEUMANN SYSTEM

Neumann (1859) posed and solved the equations of uncoupled harmonic oscillators, $\ddot{x}_j = E_j x_j$, constrained to move on the unit sphere $\sum x_j^2 = 1$ in configuration space. The equations, derived from elementary mechanics, are

$$\ddot{x}_j + q x_j = E_j x_j, \quad j = 1, \ldots, g+1, \tag{5}$$

subject to

$$\sum_{j=1}^{g+1} x_j^2 = 1, \quad \sum_{j=1}^{g+1} x_j \dot{x}_j = 0, \tag{6}$$

where q is the constraining force,

$$q = \sum_{i=1}^{g+1} E_i x_i^2 + \dot{x}_i^2. \tag{7}$$

The similarity between (5) and the eigenvalue problem $\psi'' + q\psi = E\psi$ was apparently first noted by Moser (1978). To conform to the standard notation, we will henceforth write t for the independent variable x: $\psi(t, p)$, $q(t)$, and so on.

PROPOSITION 4. *Let* $\mathrm{d}^2/\mathrm{d}t^2 + q(t)$ *be a finite-gap operator, and let* $E_1, \ldots, E_{g+1}$ *be a subset of the finite branch points of* $R^2 = \Pi(E - \epsilon_j)$. *There exist constants* $\rho_1, \ldots, \rho_{g+1}$ *such that the functions*

$$x_j(t) \underset{\text{def}}{=} \sqrt{\rho_j}\, \psi(t, E_j)$$

satisfy $\sum_{j=1}^{g+1} x_j^2(t) \equiv 1$.

Remark. Differentiate $\sum x_j^2 \equiv 1$ to get $\sum x_j \dot{x}_j \equiv 0$. Differentiate again, and use the eigenvalue equation $\ddot{x}_j + qx_j = E_j x_j$ satisfied by $x_j(t)$ to get $q = \sum E_i x_i^2 + \dot{x}_i^2$. With this representation for $q(t)$, the eigenvalue equation is rewritten

$$\ddot{x}_j + (\sum E_i x_i^2 + \dot{x}_i^2)\, x_j = E_j x_j.$$

Hence we have proved:

COROLLARY. *The* $x_j(t), j = 1, \ldots, g+1$, *solve the Neumann system.*

Proof of proposition 4 (McKean & van Moerbeke 1975; Moser 1978, 1981; Cherednik 1978). Let τ be the hyperelliptic involution. Set $\phi(t, p) = \psi(t, \tau p)$, then ψ and ϕ are two independent solutions of $\ddot{y} + qy = Ey$, except at branchpoints where $\tau p = p$ and $\psi = \phi$. The product $\psi(t, p)\, \phi(t, p)$ is meromorphic on the curve C, since the exponentials cancel; moreover, the product $\psi\phi$ is invariant under τ, and so is a rational function of E alone. In terms of the zeros $\mu(t)$ and poles δ of ψ, we have

$$\psi(t, p)\, \phi(t, p) = \prod_{j=1}^{g+1} \frac{E - E(\mu_j(t))}{E - E(\delta_j)}.$$

Let Ω be the differential

$$\Omega = \frac{1}{2} \frac{\prod_{j=1}^{g+1} (E - E(\delta_j))}{R}\, \mathrm{d}E, \tag{8}$$

and let h be the function on C,

$$h(p) = \frac{R(p)}{\prod_{j=1}^{g+1} (E - E_j)}, \tag{9}$$

which has poles at the E_j and zeros at the remaining branchpoints $e_1, \ldots, e_g$ and at ∞.

The differential $\omega = h\psi\phi\Omega$ has simple poles at $E_1, \ldots, E_{g+1}$ and at ∞. The residue at E_j is $(\mathrm{Res}_{E_j} h\Omega)\, \psi(t, E_j)\, \phi(t, E_j)$. Set $\rho_j = \mathrm{Res}_{E_j} h\Omega$; then, since $\psi = \phi$ at branchpoints, this residue is $\rho_j \psi(t, E_j)^2$. The residue at ∞ is -1. The sum of residues of ω is zero, so $\Sigma\, \rho_j \psi(t, E_j)^2 = 1$. With $x_j \underset{\mathrm{def}}{=} \sqrt{\rho_j}\, \psi(t, E_j)$ we get the desired identity. ■

The key feature of the proof (which will reappear in greater generality in §10) is the passage from ψ to the residues of $\psi\phi h\Omega$. Note that

$$\psi\phi h\Omega = \frac{\prod_{k=1}^{g} (E - E(\mu_k(t)))}{\prod_{j=1}^{g+1} (E - E_j)}\, \mathrm{d}E;$$

if we expand this in partial fractions, we get

$$\sum_{j=1}^{g+1} \frac{x_j^2(t)}{E - E_j}\, \mathrm{d}E,$$

together with the constraint

$$\sum_{j=1}^{g+1} x_j^2 \equiv 1.$$

The (many-to-one) change of variables $\mu(t) \to x^2(t)$ amounts to the introduction of ellipsoidal coordinates and was already used by Jacobi to linearize equations; we have here translated it into the language of Baker functions.

7. The vector Baker map

This section shows how transferences (§5) are used to map C to a projective space. In the process, we again encounter the Neumann system (§6). As above, let $\{e_1, \ldots, e_g, \infty\} \cup \{E_1, \ldots, E_{g+1}\}$, be a partition of the branchpoints, and let h be the function (9) with zeros $Z + \infty = e_1 + \ldots + e_g + \infty$ and poles at the E_j.

Lemma. *The functions*

$$X_j(t, p) = \sqrt{\rho_j}\, \frac{W(\psi(t, p), \psi(t, E_j))}{h(p)\,(E(p) - E_j)}, \tag{10}$$

$j = 1, \ldots, g+1$, *have poles in* $\delta + Z$, *and 'go' like* $\sqrt{\rho_j}\, \psi(t, E_j)\, \mathrm{e}^{kt} \underset{\text{def}}{=} x_j\, \mathrm{e}^{kt}$ *at* ∞ (*i.e. they are sections of* $\mathscr{L}(\delta + Z) \otimes \mathscr{L}(\mathrm{e}^{kx})$).

Proof. This is immediate from proposition 1 of §5. ∎

We look at the χ_j either as components of a $\mathbb{C}^{g+1}$-valued function on C,

$$\chi(t, p) \underset{\text{def}}{=} (\chi_1(t, p), \ldots, \chi_{g+1}(t, p))^{\mathrm{T}}, \tag{11}$$

or as homogeneous coordinates of a $\mathbb{P}^g$-valued map,

$$\hat{\chi}(t, p) \underset{\text{def}}{=} [\chi_1(t, p) : \ldots : \chi_{g+1}(t, p)]. \tag{12}$$

The circumflex distinguishes between these interpretations. We call either one the *vector Baker map*.

Just as the scalar Baker function ψ solves $\mathrm{L}\psi = E\psi$, $\mathrm{B}\psi = R\psi$, so the vector χ is a simultaneous eigenvector of two *matrix* operators $\mathbb{L}$ and $\mathbb{B}$.

Proposition 5 (Flaschka 1984). *There exist* $(g+1) \times (g+1)$ *matrices* $\mathbb{L}, \mathbb{B}$ *whose entries depend on the coordinates* x_j, $y_j = \dot{x}_j (j = 1, \ldots, g+1)$ *of the Neumann system and* (*polynomially*) *on the function* h, *such that*

$$\dot{\chi} = \mathbb{B}\chi, \tag{13}$$

$$\mathbb{L}\chi = -h^2 E\chi. \tag{14}$$

The compatibility condition between (13), (14),

$$\dot{\mathbb{L}} = [\mathbb{B}, \mathbb{L}], \tag{15}$$

is equivalent to the Neumann system.

Remarks. (1) Equation (15) is the Lax representation for the Neumann system first found by Adler & van Moerbeke (1980). It was derived from the Hill equation in Flaschka (1984), where the proof of proposition 5 may be found.

(2) The explicit form of $\mathbb{L}, \mathbb{B}$ will not be needed later. The point to note is that the process of mapping C to a projective space has produced a vector Baker function χ and a Lax pair $\mathbb{L}, \mathbb{B}$; these may be thought of as a *representation* of the scalar Baker function ψ and the scalar Lax pair L, B.

(3) By §5, $\psi(t, p)$ is a section of $\mathscr{L}(\delta) \otimes \mathscr{L}(\mathrm{e}^{kt})$ and $\chi(t, p)$ is a vector of sections of $\mathscr{L}(\delta) \otimes \mathscr{L}(\mathrm{e}^{kt}) \otimes \mathscr{L}(Z)$. At the level of Jacobians, the motions in t are therefore identical.

(4) One could map C to a $\mathbb{P}^m$ by adding to δ a divisor Z other than $e_1 + \ldots + e_g$. The degree of Z must be greater than or equal to g if matrix eigenvalue problems (13), (14) are to be obtained (Flaschka 1984). Our choice of Z was further restricted by the desire to get the usual Neumann system.

8. Rational normal scrolls

We now turn to an investigation of the geometry of the image of C under the vector Baker map χ in (12). As t changes, so does the linear system of $\mathscr{L}(\delta+Z) \otimes \mathscr{L}(e^{kt})$, and the image $\chi(t, C)$ will move in $\mathbb{P}^g$. We have not found this to be a good way of thinking about t-dependence, so we will, for now, *fix t at* $t = 0$, and abbreviate $\chi(0, p)$ to $\chi(p)$, etc. We denote the image $\chi(C)$ of C by C'.

(a) *First properties of the curve C′*

Definition. The *degree* of C' in $\mathbb{P}^g$ is the number of points of intersection of C' with a generic hyperplane in $\mathbb{P}^g$.

Lemma. *C' has degree $2g$.*

Proof. Refer to the end of §4: the intersections of a hyperplane with C' are the zeros of some function in $L(\delta+Z)$. The typical function has $2g$ poles $\delta+Z$, and hence also $2g$ zeros. ∎

The next result concerns the faithfulness of the image: is it a singularity-free copy of C?

Proposition 6. (i) *When $g = 2$, C' is a nodal quartic or a double conic in $\mathbb{P}^2$*; (ii) *When $g > 2$, C' is non-singular whenever δ lies off a certain two-dimensional subvariety of $Pic^{(g)}(C)$.*

Remark. A 'nodal quartic' is a plane curve with one node (a point when two branches of the curve cross), with the equation of the curve being a fourth-degree polynomial. The lemniscate is the most familiar example. A 'double conic' consists of two superimposed copies of a conic, for example $(x^2+y^2-1)^2 = 0$.

Proof. The image C' will be a non-singular copy of the curve if (i) $\chi(p)$ is never $[0: \ldots : 0]$ (if the linear system $L(\delta+Z)$ has no 'basepoints'); (ii) if $\chi(p_1) \neq \chi(p_2)$ when $p_1 \neq p_2$ (otherwise, there will be a node); (iii) if $d\chi(p) \neq 0$ (otherwise, there will be a cusp).

$L(\delta+Z)$ has no base points since $\deg(\delta+Z) = 2g$. To verify (ii), (iii), one must show that $\dim L(\delta+Z-p_1-p_2) = g-1$ for $p_1, p_2 \in C$ (Hartshorne 1977). By Riemann–Roch, this will be the case if there is no differential with zeros $\delta+Z$ and poles p_1, p_2. Now, since Z is non-special, there is a unique differential $\omega_{p_1 p_2}$ with zeros including Z, poles p_1, p_2, and residues ± 1 if $p_1 \neq p_2$. The differential $\omega_{p_1 p_2}$ has g zeros besides Z; call them $W_{p_1 p_2}$. $\{p_1, p_2\} \rightarrow W_{p_1 p_2}$ maps $(C \times C)_{\text{sym}}$ to a two-dimensional subvariety V of $Pic^{(g)}(C)$.

When $g = 2$, V is dense in $Pic^{(2)}(C)$, almost all δ belong to V, and there will be a node. A little more argument is needed to exclude the possibility of cusps and to see that double conics will arise; we omit this. When $g > 2$, almost all δ are not in V, and $\chi: C \rightarrow C'$ will be an embedding. ∎

(b) *Rational normal scrolls*

The curve $C' \subset \mathbb{P}^g$ lies on a very special two-dimensional surface in $\mathbb{P}^g$, whose description will take up the rest of this section.

The hyperelliptic involution $\tau: (E, R) \rightarrow (E, -R)$ on C is inherited by C', where we call it τ'. For each $P \in C'$, draw the line $\mathbb{P}^1$ through P and $\tau' P$; at branchpoints where $P = \tau' P$, draw the tangent to C'. This so-called *hyperelliptic secant* through $\chi((E, R))$ and $\chi((E, -R))$ is associated uniquely with $E \in \mathbb{P}^1$. Hence, we have a family of disjoint $\mathbb{P}^1$s indexed by $\mathbb{P}^1$; this is called a *rational ruled surface.*

Abstract rational ruled surfaces are classified by the non-negative integers and are denoted by S_n. S_0 is $\mathbb{P}^1 \times \mathbb{P}^1$, S_1 is $\mathbb{P}^2$ blown up at one point. The rational ruled surface generated by

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the hyperelliptic secants of C' is one of these two simple types. All S_n can be embedded in a canonical way in $\mathbb{P}^{n+2k+1}$; the resulting surfaces are called *rational normal scrolls* and are denoted by $S_{k,n}$ (see Griffiths *et al.* 1978). The scrolls arising in our example turn out to be of type $S_{(g-1)/2,0}$ if $g > 1$ is odd, and of type $S_{(g-2)/2,1}$ if g is even. Before giving some of the details, we want to explain in general terms the reasons for our interest in these surfaces.

First, the geometry of rational normal scrolls is very attractive, and we think it remarkable that all the basic ingredients (Griffiths & Harris 1978) in the theory of these surfaces can be calculated explicitly and naturally in terms of Baker functions. It is plausible that *all* the $S_{k,n}$ can be constructed in a uniform way from Baker functions of higher order operators L, but that has yet to be worked out. More significantly, the scrolls seem to play a role in the geometry of Kummer varieties. We do not know the whole story yet; another small result is given in the next section in connection with a discussion of the Kummer surface.

Our analysis of the rational normal scroll starts not with the embedded curve $C' \subset \mathbb{P}^g$, but with the $\mathbb{P}^2$ in which the original curve $C: R^2 = \Pi(E-\epsilon_j)$ is situated. We map all of $\mathbb{P}^2$ birationally into $\mathbb{P}^g$ in such a way that the image is the desired rational normal scroll; the curve $C \subset \mathbb{P}^2$ is carried along, and the mapping of $\mathbb{P}^2$ restricted to C is precisely the vector Baker map χ.

The map $\mathbb{P}^2 \rightarrow \mathbb{P}^g$ will be defined by a linear system on $\mathbb{P}^2$, with some variations on the general scheme of §4: the functions can be given explicitly, and there are *basepoints*, namely points in $\mathbb{P}^2$ where all functions vanish. The main desideratum, as in §7, is that the linear system have some relation to the pole divisor δ (on C) of the Baker function. Burchnall & Chaundy (1928) once again provide the answer. In the course of the proof of proposition 3, they constructed a linear space of polynomials on $\mathbb{P}^2$ whose zeros on the curve C include the zero divisor $\mu(t)$ of ψ. We first adapt their construction to our needs, and then outline the geometry of the map into $\mathbb{P}^g$.

The Burchnall–Chaundy system

Fix a Baker function $\psi(t,p)$ with poles δ and zeros $\mu(t)$, and let $\phi(t,p) = \psi(t,\tau p)$ as in §6. Let $\tilde{\psi}$ be the Baker function obtained after $g+1$ transferences at $E_1, \ldots, E_{g+1}$; write $\tilde{\delta}, w(t)$ for its poles and zeros. Set $\tilde{\phi}(t,p) = \tilde{\psi}(t,\tau p)$, and $\tilde{\psi}^{(j)} = \mathrm{d}^j\tilde{\psi}/\mathrm{d}t^j$.

The functions $\tilde{\phi}\tilde{\psi}^{(j)}$ are meromorphic on C (the exponentials at ∞ cancel) and have finite poles $\tilde{\delta}_j + \tau\tilde{\delta}_j$. If $r(p) \underset{\text{def}}{=} \Pi_{j=1}^{g}(E(p)-E(\tilde{\delta}_j))$, therefore, $r\tilde{\phi}\tilde{\psi}^{(j)}$ will have no finite poles.

LEMMA. *There is a linear combination of* $r\tilde{\phi}\tilde{\psi}, r\tilde{\phi}\tilde{\psi}^{(1)}, \ldots, r\tilde{\phi}\tilde{\psi}^{(g+1)}$ *that vanishes precisely at* $\tau w(t)$, $\mu(t)$, *and* $E_1 + \ldots + E_{g+1}$.

Proof. The term $r\tilde{\phi}\tilde{\psi}^{(g+1)}$ has a pole of order $3g+1$ at ∞, so the general linear combination has $3g+1$ zeros. We claim that

$$r\tilde{\phi}W(\tilde{\psi}(t,p), \tilde{\psi}(t,E_1), \ldots, \tilde{\psi}(t,E_{g+1})), \tag{16}$$

(where W is the $(g+2)\times(g+2)$ Wronskian determinant) has the desired property. The expression (16) is clearly a linear combination of the required form, and $\tilde{\phi}$ vanishes at $\tau w(t)$, while W vanishes at $E_1 + \ldots + E_{g+1}$. To see that W also vanishes at $\mu(t)$, note first that the Baker function obtained from $\tilde{\psi}$ by transference at $E_1, \ldots, E_{g+1}$ is just the original ψ. Indeed, by proposition 3, $A(\tilde{\delta}) = A(\delta) + A(E_1) + \ldots + A(E_{g+1})$; a repeated transference at those same points gives $A(\delta) + 2A(E_1) + \ldots + 2A(E_{g+1})$. But $2A(E_j) \equiv 2A(\infty) \equiv 0$, since $2E_j \equiv 2\infty$, and ∞ is the basepoint for the Abel map. Now, W is not quite that twice transferred Baker function; to get the original ψ, divide W by $W(\tilde{\psi}(t, E_1), \ldots, \tilde{\psi}(t, E_{g+1}))$ and call the result $\xi(t, p)$. Then $\psi(t, p) = \xi(t, p)/\xi(0, p)$. Note that the zeros $E_1 + \ldots + E_{g+1}$ of W are cancelled in this procedure, but the other g zeros are unaffected. They must therefore be just the zeros $\mu(t)$ of ψ. ■

This is the key part of the Burchnall–Chaundy construction. There are now just a few mechanical steps left before we arrive at the desired linear system on $\mathbb{P}^2$.

(1) Set $t = 0$ (cf. the beginning of this section); then $\tilde{\phi} \equiv \tilde{\psi} \equiv 1$, and the $\tilde{\psi}^{(j)}$ are meromorphic on C. The functions (16) assume the simple form: $r, r\tilde{\psi}^{(1)}, \ldots, r\tilde{\psi}^{(g+1)}$.

(2) The $r\tilde{\psi}^{(j)}$ have no finite poles on C, and so are polynomials in E and R. They therefore make sense on $\mathbb{C}^2$ (the (E, R)-plane) and not just on C.

(3) Because $\ddot{\tilde{\psi}} = (E-q)\,\tilde{\psi}$, $\tilde{\psi}^{(j)}$ for $j \geqslant 2$ can be expressed in terms of $\tilde{\psi} \equiv 1$, $\tilde{\psi}^{(1)}$, and powers of E. Namely, $\tilde{\psi}^{(j)} = p_j(E) + q_j(E)\,\tilde{\psi}^{(1)}$, for certain polynomials p_j, q_j, and so

$$r\tilde{\psi}^{(j)} = rp_j(E) + rq_j(E)\,\tilde{\psi}^{(1)}. \tag{17}$$

(4) It is easy to calculate $r\tilde{\psi}^{(1)}$ explicitly. The function $\tilde{\psi}^{(1)}$ is the unique meromorphic function with poles $\tilde{\delta} + \infty$ and behaviour k + lower order terms at ∞. The result is

$$r\tilde{\psi}^{(1)}(E, R) = R + \sum_{j=1}^{g} c_j \prod_{i(\neq j)} (E - E(\tilde{\delta}_j)); \tag{18}$$

the c_j are certain constants, chosen so that the polynomial vanishes at $(E(\delta_j), R(\tau\delta_j))$, i.e. at $\tau\tilde{\delta}_j$ on C.

(5) Extend the polynomials from $\mathbb{C}^2$ to $\mathbb{P}^2$: set $E = \hat{E}/\hat{U}$, $R = \hat{R}/\hat{U}$, and clear powers of $\hat{U}$ from the denominator.

(6) After writing out a few of the projectivized polynomials $r\tilde{\psi}^{(j)}$, $j = 0, \ldots, g+1$, one finds a simple structure corresponding to (17). We now summarize these results.

Definition. Set

$$G_1 = \prod_{j=1}^{g} (\hat{E} - E(\tilde{\delta}_j)\,\hat{U})$$

(the projectivized r), and

$$G_2 = \hat{U}\left\{\hat{U}^{g-2}\hat{R} + \sum_{j=1}^{g} c_j \prod_{i(\neq j)} (\hat{E} - E(\hat{\delta}_j)\,\hat{U})\right\}$$

(cf. equation (18)). Let $m = \frac{1}{2}(g-1)$ if $g > 1$ is odd, $m = \frac{1}{2}g$ if g is even. The *Burchnall–Chaundy system on* $\mathbb{P}^2$ is the linear span $\mathscr{S}$ of the products

$$\{\hat{U}^m, \hat{E}\hat{U}^{m-1}, \ldots, \hat{E}^m\} \otimes \{G_1, G_2\} \tag{19}$$

if g is odd, and the linear span of these products with $\hat{E}^m G_2$ deleted, if g is even. ($\mathscr{S}$ depends on the pole divisor δ of ψ.)

Proposition 7. *The linear space $\mathscr{S}$ has dimension $g+1$ (this reveals a hidden linear dependence among the functions* (16)). *All functions in $\mathscr{S}$ vanish at the finite basepoints $(E(\tilde{\delta}_j), R(\tau\tilde{\delta}_j))$. There are further basepoint conditions at infinity, $[\hat{U}:\hat{E}:\hat{R}] = [0:0:1]$, which we will not spell out. $\mathscr{S}$ restricted to C is the vector Baker system $L(\delta + Z)$ of* §7.

The map of $\mathbb{P}^2$ into $\mathbb{P}^g$

The system (19) contains two pencils of rational curves: $G_\beta = \beta_1 G_1 + \beta_2 G_2$, $[\beta_1:\beta_2] \in \mathbb{P}^1$, and $L_\alpha = \alpha_1 \hat{E} + \alpha_2 \hat{U}$, $[\alpha_1:\alpha_2] \in \mathbb{P}^2$. (The latter are the vertical lines $E = \text{const.}$) Now blow

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up $\mathbb{P}^2$ at the basepoints until the map is well defined, and blow down the contractible cross-fibres. The details are not hard, but the 'book-keeping' would be too much to reproduce. On the surface $\tilde{\mathbb{P}}^2$ eventually obtained from $\mathbb{P}^2$, the proper transforms $\tilde{L}_\alpha$ and $\tilde{G}_\beta$ of L_α, G_β form a rational double ruling: $\tilde{L}_\alpha^2 = \tilde{G}_\beta^2 = 0$, $\tilde{L}_\alpha . \tilde{G}_\beta = 1$. $\tilde{\mathbb{P}}^2$ is therefore isomorphic to $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

The system $\mathscr{S}$ by which S_0 is embedded in $\mathbb{P}^g$ is the linear system $L(m\tilde{L}+\tilde{G})$, which shows (Griffiths *et al.* 1978) that the image is $S_{k,0} = S_{(g-1)/2,0}$ if g is odd, $= S_{g/2,0}$ if g is even. When g is odd, we have found the desired scroll, since (19) is the Burchnall–Chaundy system. When g is even, the Burchnall–Chaundy system is (19) with one basis element removed. That creates another basepoint condition at ∞, which is resolved by blowing up and down one more time in just the way that takes $S_{g/2,0}$ to $S_{(g-2)/2,1}$ (Griffiths *et al.* 1978, p. 527).

The main point to appreciate is this: *the Burchnall–Chaundy system of polynomials built from Baker functions provides exactly the rational pencils and linear systems on $\mathbb{P}^2$ that produce the rational normal scrolls from $\mathbb{P}^2$ according to the classical method.*

9. The Kummer surface, quadrics, and the vector Baker plane

So far we have fixed a pole divisor δ of ψ, set $t = 0$, and studied the vector Baker map of the curve C into $\mathbb{P}^g$. We next want to ask how this picture changes as δ varies over $Pic^{(g)}(C)$. A clue is furnished by the classical theory of Kummer's quartic surface in $\mathbb{P}^3$ (Hudson 1905).

Suppose C has genus 2. There is a familiar map of $Jac(C)$ into $\mathbb{P}^3$, given, according to §4, by a certain linear system on $Jac(C)$. Linear systems on Jacobians are built from θ-functions. This is not a suitable place to review θ-functions; we just remind the reader of the functional equations that determine the different types of θ-functions. We take $Jac(C) = C^g/\Lambda$, with period lattice $\Lambda = \mathbb{Z}^g + \Omega\mathbb{Z}^g$; Ω is a symmetric $g \times g$ matrix, $\operatorname{Im}\Omega > 0$. The definitions hold for arbitrary g.

(1) *Theta function.* $\theta(z+m) = \theta(z)$,

$$\theta(z+\Omega m) = \exp(-i\pi\Omega m \cdot m - 2i\pi m \cdot z)\,\theta(z), \quad z \in \mathbb{C}^g, \quad m \in \mathbb{Z}^g. \tag{20}$$

(2) *Theta functions of order r.* Instead of (20),

$$f(z+\Omega m) = [\exp(-i\pi\Omega m \cdot m - 2i\pi m \cdot z)]^r f(z).$$

(3) *Theta function with characteristic.* For $a, b \in \mathbb{C}^g$,

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z) = \exp(i\pi\Omega a \cdot a + 2i\pi a \cdot (z+b))\,\theta(z+\Omega a + b).$$

Below, we write $\theta(z, \Omega)$ to indicate the dependence of θ on the Riemann matrix Ω.

Returning now to the genus 2 Jacobian $Jac(C)$, we consider the map to projective space given by second-order ('quadratic') θ-functions. It is a classical result (Mumford 1983) that the linear space of quadratic θ-functions has the basis $\theta\begin{bmatrix} \frac{1}{2}a \\ 0 \end{bmatrix}(2z, 2\Omega)$, $a = (a_1, a_2) \in \mathbb{C}^2$, $a_i = 0, 1$. Let $\theta_1, \theta_2, \theta_3, \theta_4$ be a labelling of these functions. Even though the θ_j are not well defined on $Jac(C)$ because of the factor of automorphy, exp (...), in (20), the map $T: z \to [\theta_1(z) : \ldots : \theta_4(z)]$ makes sense projectively (a common factor is irrelevant).

T is a 2-to-1 map from $Jac(C)$ into $\mathbb{P}^3$. $T(z_1) = T(z_2)$ when z_1 and z_2 are related by the hyperelliptic involution, that is, when $z_1 = A(\nu_1+\nu_2)$, $z_2 = A(\tau\nu_1+\tau\nu_2) = -A(\nu_1+\nu_2)$. The

image $T(Jac(C)) \underset{\text{def}}{=} S$ is known to be defined by a fourth-degree polynomial in $\mathbb{P}^3$; it is called the *Kummer surface*.

A tangent plane to the surface S cuts S in a curve; because S is quartic, this curve has degree 4 in the tangent plane. It is birational to the original genus 2 curve C; indeed, it is a *nodal quartic* (cf. §8) in the tangent plane $\mathbb{P}^2$, and the tangency point is at the node (Hudson 1905).

These classical results suggest the following picture. For each $\delta \in Pic^{(g)}(C)$, the vector Baker map sends the hyperelliptic curve C to a curve C' in a $\mathbb{P}^g$ that is tangent to a 2-to-1 image in $\mathbb{P}^{g+1}$ of $Jac(C)$, and as δ varies over $Pic^{(g)}(C)$, the tangent hyperplanes envelop this 2-to-1 copy of the Jacobian.

This result is proved in the next section, unfortunately in a non-constructive way. To illustrate the close connection between quadratic θ-maps and our vector Baker embedding, we close this subsection by rediscovering the scrolls of §8. Let $T: Jac(C) \to \mathbb{P}^{2^g-1}$ be the quadratic θ-map of the Jacobian of C, with genus now greater than or equal to 2. The Abel map A embeds C in $Jac(C)$ (cf. §3). Let $\tilde{C} = T \circ A(C)$.

Proposition 8. *$\tilde{C}$ is a curve of degree $2g$ contained in a $\mathbb{P}^g \subset \mathbb{P}^{2^g-1}$. It lies on a rational normal scroll of the kind described in* §8.

Sketch of proof. T is the map given by the complete linear system $|2\theta|$, where θ is the theta divisor. By Jacobi inversion, 2θ intersects $A(C)$ in $2g$ points, which are linearly equivalent to a divisor of the form $\delta + Z$. So $T|_{A(C)}$ is just a vector Baker embedding. ■

The vector Baker plane in $\mathbb{P}^{g+1}$

In §7, we mapped C to $C' \subset \mathbb{P}^g$ via $\tilde{\chi}$. To be able to think of these $\mathbb{P}^g$s as enveloping a hypersurface, we must put them into a $\mathbb{P}^{g+1}$. This construction is given next.

Definition. $\hat{\Xi}: C \to \mathbb{P}^{g+1}$ is the map

$$\hat{\Xi}(p) = [\psi(p) - h(p)\,\dot{\psi}(p) : h(p)\,\chi_1(p) : \ldots : h(p)\,\chi_{g+1}(p)]. \tag{21}$$

Note that, but for the first coordinate, this is just the vector Baker map (12). We now give some sample results to justify this definition. Let $[U_0 : \ldots : U_{g+1}]$ be the ambient coordinates in $\mathbb{P}^{g+1}$. Recall the definition of the Neumann variables:

$$x_j = \sqrt{\rho_j}\,\psi(E_j), \quad y_j = \dot{x}_j = \sqrt{\rho_j}\,\dot{\psi}(E_j); \quad \text{set } x = (x_1, \ldots, x_{g+1}), \quad y = (y_1, \ldots, y_{g+1}).$$

Proposition 9. (i) *$\hat{\Xi}(C)$ lies on the hyperplane $\hat{V} \subset \mathbb{P}^{g+1}$ with equation*

$$U_0 + y_1 U_1 + \ldots + y_{g+1} U_{g+1} = 0,$$

and $\hat{\Xi}(C) \subset \hat{V} \cong \mathbb{P}^g$ is just the vector Baker embedding (cf. §7) *of C.*

(ii) *Let $\hat{l}$ be the intersection of $\hat{V}$ with the hyperplane $x_1 U_1 + \ldots + x_{g+1} U_{g+1} = 0$. Then $\hat{l}$ intersects $\hat{\Xi}(C)$ in $\delta + Z$.*

(iii) *Let $\hat{\xi}$ be the line in $\mathbb{P}^{g+1}$ dual to the $(g-1)$-plane $\hat{l}$. Then $\hat{\xi}$ is simultaneously tangent to the confocal quadrics*

$$\sum_{j=1}^{g+1} \frac{U_j^2}{e_k - E_j} = U_0^2, \quad k = 1, \ldots, g. \tag{22}$$

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Sketch of proof. (i) By comparing behaviour at poles and at ∞, one checks that

$$\sum_{j=1}^{g+1} y_j\, h(p)\, \chi_j(p) = h(p)\, \dot{\psi}(p) - \psi(p).$$

(ii) In the same way, check that $\psi(p) = \sum_{j=1}^{g+1} x_j \chi_j(p)$. So the points on $\hat{l} \subset \hat{\Xi}(C)$ are the zeros of h and the zeros of ψ; when $t = 0$, the latter are located at δ.

(iii) $\hat{\xi}$ is the projectivization of the affine line $\{y+sx \mid s \in \mathbb{C}\}$. It is a common tangent to the quadrics (22) by results of Moser (1981) and Knörrer (1980).

For the benefit of the reader familiar with Moser (1981), we remark that $\hat{V}$ is the g-plane spanned by the (projectivized) gradients to the Q_k at the points of tangency of $y+sx$, and by the projectivization of x. That is, *$\hat{V}$ is spanned by the projective Chasles frame*. The line $\hat{l}$ is spanned in $\hat{V}$ by the gradients only.

The rather appealing conclusion is that *the Neumann pole divisor $\delta + Z$ is realized as the $(g-1)$-plane dual to a common tangent to the confocal quadrics* (22). We omit a detailed discussion, to progress to some newer geometric ideas in the next section.

10. Intersections of quadrics

In this final section, we continue the discussion of §9 in a new direction. The constructions are motivated by two goals. From the viewpoint of *soliton theory*, one wants to see a solvable system, such as Neumann's, as a flow on a geometric object, in a way that makes its 'hidden symmetries' as obvious as possible. This approach goes back to Lie, who studied separable Hamilton–Jacobi equations with Monge cones defined by line complexes. We want to generalize Lie's theory to $g > 2$, but since our work is just beginning, we will not stress this point of view. The *geometric* problem addressed in this section is to construct an extrinsic and manageable, but not necessarily faithful, representation of the set of all effective divisors of given degree d on a curve C. We will first describe this problem in outline, and then show how Baker functions and transferences enter in a natural way.

(a) *Sets of divisors*

Let C be a not necessarily hyperelliptic curve, and for $d > 0$ let $C^{(d)}$ be the set of positive divisors of degree d (*not* the set $Pic^{(d)}(C)$ of linear equivalence classes!). $C^{(d)} \cong C^d/S_d$ = set of unordered d-tuples of points of C; $C^{(d)}$ is an algebraic variety as a result of this identification. Complete linear systems of divisors of degree d (i.e. linear equivalence classes) consist precisely of the maximal (projective) linear subspaces of $C^{(d)}$. The maximal linear subspaces are not necessarily all of the same dimension; this is the case if, and only if, $d \geqslant 2g-1$. When $d \geqslant 2g-1$, all divisors are non-special, and by Riemann–Roch the dimension of the linear system is $d-g+1$. Projectively, the dimension is $d-g$. We therefore picture $C^{(d)}$ as a space foliated by $\mathbb{P}^{d-g}$s, each of which consists of a maximal set of divisors linearly equivalent to each other. Alternatively, we may start with the set of linear equivalence classes of degree d divisors; it is $Pic^{(d)}(C)$, a torus. To each $[\alpha] \in Pic^{(d)}(C)$, we attach $\mathbb{P}^{d-g} \cong \mathbb{P}L(\alpha)$, the set of divisors linearly equivalent to α. Thus, $C^{(d)}$ can also be viewed as a fibre bundle over $Pic^{(d)}(C)$.

We are interested in our hyperelliptic C, and in the set $C^{(2g)}$ of effective divisors of degree $2g$. A model for $C^{(2g)}$ will be described in subsection 10(c). For further background on the general question, see Griffiths (1976).

(b) *Cell structure on linear systems*

Geometric objects are particularly pleasant to analyse if they have a cell decomposition. They are then written as disjoint union of Euclidean spaces (of varying dimensions). $\mathbb{P}^g$ has a natural cell decomposition:

$$\mathbb{P}^g = \mathbb{C}^g \cup \mathbb{C}^{g-1} \cup \ldots \cup \mathbb{C} \cup \{\text{point}\}.$$

In homogeneous coordinates, $\mathbb{C}^g = \{[U_0: \ldots: U_g] \mid U_0 \neq 0\}$, $\mathbb{C}^{g-1} = \{[0: U_1: \ldots: U_g] \mid U_1 \neq 0\}$, etc.

Now fix $[\beta] \in Pic^{(2g)}(C)$. Consider the projectivized linear space $\mathbb{P}L(\beta)$. Now, β is linearly equivalent to $\delta + Z$ for some δ, so we take $\delta + Z$ as the representative of the equivalence class of β. $\mathbb{P}L(\delta+Z) \cong \mathbb{P}^g$, and therefore has a cell decomposition.

Proposition 10. *The set* $M_i \underset{\text{def}}{=} \{\alpha \in C^{(2g)} \mid \alpha \equiv \delta + Z$, *and precisely* i *of the points in* α *are* $\infty\}$ *is* $\mathbb{C}^{g-i}$. $\mathbb{P}L(\delta+Z) = M_0 \cup M_1 \cup \ldots \cup M_{g-1} \cup M_g$ *is a cell decomposition of* $\mathbb{P}L(\delta+Z)$.

Proof. We will exhibit, for each $i = 0, \ldots, g$, an affine subspace V_i of $L(\delta+Z)$; each function in V_i will have its zero divisor in M_i. These V_i, isomorphic to $\mathbb{C}^{g-i}$, will coordinatize the divisor sets M_i.

Let $\psi^{(0)}$ be the Baker function with pole divisor δ. Let $\psi^{(1)}$ be the partial transference at e_1 (see equation (3)). Let $\psi^{(2)}$ be the partial transference of $\psi^{(1)}$ at e_2,

$$\psi^{(2)}(p) = \frac{W(\psi^{(1)}(p), \psi^{(1)}(e_2))}{E(p) - e_2}$$

(we set $t = 0$), and define $\psi^{(3)}, \ldots, \psi^{(g)}$ by continued partial transferences at $e_3, \ldots, e_g$. Each $\psi^{(i)}$ is in $L(\delta+Z)$. One sees easily that $\psi^{(i)}$ has a zero of order i at ∞, so the functions are independent and form a basis of $L(\delta+Z)$.

Let $V_i = \psi^{(i)} +$ linear span of $\{\psi^{(i+1)}, \ldots, \psi^{(g)}\}$. Clearly, the zero divisor of $f \in V_i$ contains ∞ precisely i times, and no f that is not a (non-zero) multiple of a function in V_i has this property. $V_i \cong \mathbb{C}^{g-i}$, as asserted. ∎

(c) *Intersection of quadrics*

We now construct a model for the complicated variety $C^{(2g)}$. This will not be a faithful projective embedding; we will sacrifice faithfulness for relative simplicity, and accept a certain amount of ambiguity as long as it seems manageable. That criterion should become clearer as we proceed with the example.

Let Ψ denote the typical function in $L(\delta+Z)$ with zero divisor in M_0 (no zeros at ∞). Given Ψ, write $\Phi(p) = \Psi(\tau p)$. The differentials (see §6) $\omega = \Psi\Phi h\Omega$ have poles at all branchpoints $\epsilon_1, \ldots, \epsilon_{2g+1}$ and at ∞; let $w_i = \mathrm{Res}_{\epsilon_j}\,\omega$, and $w_0 = -\mathrm{Res}_{\infty}\,\omega$. Since the sum of residues of ω is zero, the w_j are constrained by

$$\sum_{j=1}^{2g+1} w_j = w_0. \tag{23}$$

Because $\Psi = \Phi$ at branchpoints, the w_j are of the form

$$w_j = \rho_j\, \Psi^2(\epsilon_j);$$

ρ_j coincides with the quantity introduced in §6 when $\epsilon_j = E_j$. As in §6, we set $w_j = z_j^2$, $j = 0, \ldots, 2g+1$; z_j is determined up to $\pm$ sign.

We view the preceding steps as providing ellipsoidal coordinates on M_0 (cf. conclusion of §6). Given $\alpha \in M_0$, we map

$$\alpha \to \omega = \frac{\prod_{j=1}^{2g} (E - E(\alpha_j))}{R^2} \, \mathrm{d}E = \sum_{j=1}^{2g+1} \frac{z_j^2}{E - \epsilon_j} \, \mathrm{d}E.$$

The map $\alpha \to \{z_j^2\}$ via $\{E(\alpha_j)\}$ is 2^{2g}-to-1. For given $\{z_j^2\}$, there are 2^{2g+1} sets $\{z_j\}$; this indeterminacy is common to ellipsoidal coordinates in general, and is the ambiguity mentioned at the outset.

Modulo this ambiguity, we have a model for $C^{(2g)}$ as the set of points on the quadric $\sum_{j=1}^{2g+1} z_j^2 = z_0^2$ in $\mathbb{P}^{2g+1}$.

This much has dealt with the big cell, $\mathbb{C}^g \subset \mathbb{P}^g$. If we now do a similar construction with $\alpha \in M_1$, we find, because $\operatorname{Res}_\infty \Psi\Phi h\Omega = 0$, that

$$\sum_{j=1}^{2g+1} z_j^2 = 0. \tag{24}$$

Set $-z_0^2 = \operatorname{Res}_\infty E\Psi\Phi h\Omega$. Then $\sum \operatorname{Res} E\Psi\Phi h\Omega = 0$ gives

$$\sum_{j=1}^{2g+1} \epsilon_j z_j^2 = z_0^2. \tag{25}$$

Modulo the same ambiguity as before, we see that the finite, effective divisors on C of degree $2g-1$ ($\alpha \in M_1$ has the form $\sum_{j=1}^{2g-1} \alpha_j + \infty$) are mapped to the intersection of the quadrics (24), (25).

We can continue this procedure for $\alpha \in M_2, \ldots, M_g$; if $\alpha \in M_i$, the corresponding ellipsoidal coordinates lie in the intersection of the $i+1$ quadrics

$$\sum_{j=1}^{2g+1} \epsilon_j^k z_j^2 = 0, \quad k = 0, \ldots, i-1, \quad \sum_{j=1}^{2g+1} \epsilon_j^i z_j^2 = z_0^2. \tag{26}$$

So far we have talked only about the *points* in $C^{(2g)}$. We can also ask about the linear subspaces, $\mathbb{P}^g$, of divisors linearly equivalent to $\alpha \in C^{(2g)}$. It is easy to see that they map to linear subspaces in the quadric $\sum z_j^2 = z_0^2$, again modulo the ambiguity mentioned earlier.

In a similar way, we have a correspondence, with a finite degree of indeterminacy, between equivalence classes of non-special divisors of degrees $2g, \ldots, g$ and linear subspaces of dimension $g, g-1, \ldots, 0$ in the intersection of $0, 1, \ldots, g$ quadrics (26).

The ambiguities inherent in this arrangement are yet to be resolved. In a couple of special cases, the answers can be found in the literature.

Example 1. Reid (1971) proved that the set of $(g-1)$-planes in the intersection of the two quadrics (24), (25) is precisely the Jacobian of C. There is, therefore, a 1-to-1 correspondence between points of $Pic^{(g)}(C)$, with their $\mathbb{P}^{g-1}$s attached, and the $(g-1)$-planes in (24), (25) (these correspond to divisors $\alpha \in M_1$; the $\mathbb{P}^{g-1}$ is ∞ + the union $\mathbb{C}^{g-1} \cup \ldots \cup \mathbb{C} \cup \{\text{point}\} \cong \mathbb{P}^{g-1}$).

Example 2. When $g = 2$, the smallest cell M_2 consists of non-special divisors of degree 2. The corresponding linear systems are $\mathbb{P}^0$, i.e. consist of one point. The set of points in the intersection of three quadrics (26) is the classical $K3$ surface, known to be birational to the Kummer surface. This suggests that the intersection of $g+1$ quadrics (26) in genus g, a '$K3$ variety', should play a similar role.

Example 3. Lower-dimensional subspaces in the intersection of two quadrics are known to play a role in the construction of moduli spaces for vector bundles over hyperelliptic curves (Desale & Ramanan 1976; Atiyah 1955). We hope that the coordinatization by transferred Baker functions may make the analysis of these moduli spaces tractable, but we have no results yet.

(*d*) *More on* $\pm$, *and on the Kummer variety*

We first encountered the square root ambiguity in §6, when we set $x_j(t) = \sqrt{\rho_j}\,\psi(t, E_j)$. A very simple example shows that this $\pm$ freedom should be expected when one tries to map a finite-gap potential $q(t)$ to a Neumann system.

The simplest stationary Lax equation $[\mathrm{L}, \mathrm{B}] = 0$, for $g = 1$, leads to an equation $\dot{q}(t) =$ (cubic in $q)^{\frac{1}{2}}$, which is solved in terms of the Weierstrass $\mathscr{P}$-function. The corresponding $g = 1$ Neumann system leads to $\dot{x}_1(t) =$ (quartic in $x_1)^{\frac{1}{2}}$, which is solved by Jacobi elliptic functions. The familiar formula $(\mathscr{P} - e_i)^{-1} = sn^2$ relates the $\mathscr{P}$ of one period parallelogram $Z + \Omega Z$ to the sn of $2Z + 2\Omega Z$.

When passing from a finite-gap $q(t)$ to the Neumann solutions $x_j(t)$, one might therefore expect that the Jacobian $\mathbb{C}^g/\Lambda$ of $g(t)$ should have to be doubled. Indeed, the $x_j(t)$ flow is linearized on $C^g/2\Lambda$. This is done explicitly by Mumford (1984) (and appears to be a general feature of matrix Lax equations when the degree of the linear system involved is about double the genus of the isospectral curve). We do not yet have a way to handle this feature within a Baker-function framework, i.e. without appeal to explicit θ-function calculations, which we like to avoid.

In one special case we are able to control all ambiguities by using Reid's theorem and some of Mumford's formulas. We close by sketching the situation; our results are not as explicit as we expect them to be eventually, so we omit the details.

Consider the set of $(g-1)$-planes in the intersection of the two quadrics (24), (25); it is isomorphic to $Jac(C)$, as explained in example 1. The projection $\pi: \mathbb{P}^{2g+1} \rightarrow \mathbb{P}^{g+1}$, $\pi: [z_0 : \ldots : z_{2g+1}] \rightarrow [z_0 : \ldots : z_{g+1}]$ takes a $(g-1)$-plane $\hat{\lambda}$ in $\mathbb{P}^{2g+1}$ to a $(g-1)$-plane $\hat{l}$ in $\mathbb{P}^{g+1}$. Knörrer (1980) shows that such an $\hat{l}$ has the properties described in proposition 9 (iii): its dual is a common tangent to the quadrics (22). In the language of §9, $\hat{l}$ is contained in a unique $\hat{V}$. The vector Baker plane $\hat{V}$ has normal $[-1 : y_1 : \ldots : y_{g+1}]$. As $\hat{\lambda}$ varies over $(g-1)$-planes in the intersection of the two quadrics (24), (25), the planes $\hat{V}$ envelop an algebraic variety. Send $y_j \rightarrow y_j^2$. One can show that the map

$$\text{Jacobian} = \{\hat{\lambda} \text{ in the two quadrics}\} \rightarrow \hat{V} \rightarrow (\text{image of enveloped variety under } y_j \rightarrow y_j^2)$$

is precisely 2-to-1; its image is the Kummer variety (the image of $Jac(C)$ under the quadratic θ-map) realized as a hypersurface in $\mathbb{P}^{g+1}$. We still must find a way to generate the polynomial defining it; soliton theory provides many identities, which, we hope, will be of use.

11. Conclusion

We have tried to show that Baker functions and transferences are marvellously suited for explicit geometric constructions. They map $\mathbb{P}^2$ to rational normal scrolls just as the textbooks prescribe. They provide flags that generate natural cell decompositions of projectivized linear systems. They define ellipsoidal coordinates for sets of effective divisors.

Obviously, the ideas in the last two or three sections need to be polished. After that is done, we plan to turn to non-hyperelliptic situations. The Burchnall–Chaundy theory is quite general, and the Neumann system constructions of §§6, 7 have been adapted to the so-called n-gonal curves by Schilling (1982, 1984). It is not unreasonable to hope for a geometric realization of the Jacobian of a trigonal curve in the manner of Reid. Meanwhile, the geometry of hyperelliptic soliton theory presents many attractive problems.

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